This systems analysis problem pertains to the system block diagram below. There is a controller that consists of a proportional constant $K_P$, a power amplifier constant $K_A$, a mechanical system consisting of a rolling cart (mass, $m$), rolling viscous friction (damping, $b$) and a retro-reflective position sensor (optical or sonic). The scale of the position sensor is represented by $K_H$ in the feedback loop.

**a.** Write the equation of motion for the rolling mass in terms of input force $f(t)$ and output displacement $y(t)$, where $y(t) = 0$ corresponds to the initial starting position of the mass. *(4 points)*

**b.** Find the expression for output displacement with respect to input force, $Y(s)/F(s)$, in the frequency domain (the $s$ plane). This is your plant, $G$, in the diagram below. *(8 points)*

**c.** Using your function for $G$, find the closed-loop transfer function, $T_1$, for the system above. *(5 points)*

**d.** If input $R = A/s$ (step input), find the value of $K_H$ in terms of $A$ for a steady-state displacement of $y_{ss} = 0.2$ meters. *(5 points)*

**e.** If $K_P = 1$, $K_A = 1$, $K_H = 1$, $m = \frac{1}{2}$ kg, and $b = 1$ Ns/m, write the closed-loop transfer function, $T_1$, in standard form. *(4 points)*
f. Find the roots of the characteristic equation of $T_1$. (4 points)

g. Find the values for the damping ratio ($\xi$) and the undamped natural frequency ($W_n$) for $T_1$. (4 points)

$$\xi = \quad W_n =$$

h. Draw in your solution the pole-zero plot for $T_1$, and include/label as many parameters as possible. (6 points)

i. For $T_1$, find the expression for $y(t)$ when $r(t) = u(t)$, the unit step function. (2 points)

j. For $T_1$, estimate the percent overshoot, time-to-peak, and 2% settling time for a step input ($r(t) = u(t)$). (6 points)

$$PO = \quad T_p = \quad T_s =$$

k. Sketch in your solution the response to a unit step input ($1/s$) to $T_1$, and include/label as many parameters as possible. (5 points)
Problems

1. Now add another negative feedback loop that differentiates the output $y$ and multiplies the result by a constant, $K_D$, as shown below. For your same plant, $G$, from above, if $K_P = 2$, $K_A = 1$, $K_H = 1$, and $K_D = 1$, write the closed-loop transfer function, $T_2$, in standard form. (10 points)

![Feedback Loop Diagram]

m. Find the roots of the characteristic equation of $T_2$. (4 points)

n. Find the values for the damping ratio ($\xi$) and the undamped natural frequency ($W_n$) for $T_2$. (4 points)

$$\xi =$$

$$W_n =$$

o. Draw in your solution the pole-zero plot for $T_2$, and include/label as many parameters as possible. (4 points)

![Pole-Zero Plot]

p. Sketch in your solution the response to a unit step input ($1/s$) to $T_2$, and include/label as many parameters as possible. (4 points)

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q. Compare/contrast T₁ and T₂. (4 points)

r. A new dynamic control element is added to the original system, as shown below. As before, $K_A = 1$, $K_H = 1$, and within the plant, $m = \frac{1}{2}$ kg, and $b = 1$ Ns/m. But now, $K_P = 8.4$. The control element has the form:

$$C = \frac{2(s + 8)}{8(s + 2)}$$

Investigate the relative stability of the system by sketching in your solution the appropriate Bode amplitude and phase responses. Describe the system’s stability. Support your answer using characteristics of the Bode plots. (13 points)
s. What type of controller is $C$? (2 points)

t. For a step input, determine the steady state error for the control system shown in part r. Support your answer. (4 points)