The following differential equation is a familiar sight to any undergraduate engineering student:

\[ m\ddot{y} + ky = 0 \]  

where \( k \) is any non-zero constant.

The well known solution, often introduced as “and a good guess to solve this equation is…”

\[ y(t) = A\sin(\omega t) + B\cos(\omega t) \]  

where \( \omega = \sqrt{\frac{k}{m}} \).

The question is then: If you’re not a great guesser, how do you find the solution to (1)?

Well, let’s take a crack at a series solution – i.e. represent the solution as a power series with correctly chosen coefficients.

Part 1 (10 points)

Start with the power series:

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]  

- Write down the appropriate derivatives of \( y \) and plug them back into the equation.
- Simplify the resulting equation keeping ONLY non-zero terms.

Part 2 (20 Points)

Take the resulting equation from Part 1 and make sure that all of the terms are under one series (hint: there are two parts to this step – get the indices, \( n \), of all summations to start at the same value and second, get the powers of \( x \) to be the same using a nifty substitution like \( m = n \pm a \) where \( a \) is the integer in YOUR equation.

Part 3 (30 Points)

You now should have one summation of some number of terms = 0. In order for this to be true, every term must be zero, so you can cast away the summation and get a thing called the
recurrence formula – that is the higher order coefficients as a function of the lower order coefficients.

- Write this relationship here
- Write the first 5 higher order coefficients $a_2$-$a_5$ in terms of $a_0$ ad $a_1$:

Part 4 (20 Points)

Now that you have all of the coefficients $a_n$, you can write down the solution as a series solution using Equation (1). Write the solution here....

Part 5 (20 Points)

Ok – so that was a lot of work, how does that give us the solution, Equation (2), that was a “good guess”?

Hint:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$