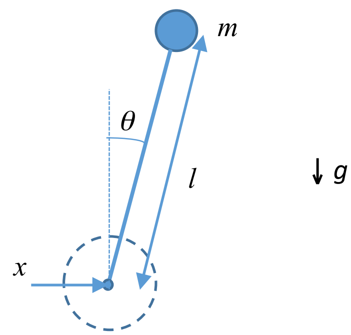
Segway is a very innovative self-balancing transportation vehicle. It is actually an inverted pendulum. For simplicity, let us only consider a **single DOF system *with displacement x as input***, as shown in the following figure, where *m* is the concentrated mass, *l* is the distance of the mass to the rotation point, and the gravity constant is *g*.

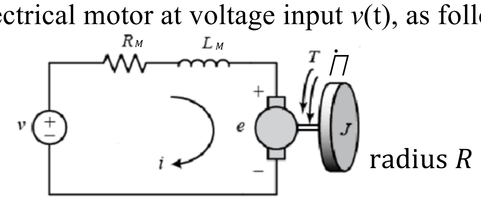


1) Inverted Pendulum System Modelling:
   a. (10 points) Please write the dynamics equation of θ(t) with the input x(t).
   b. (10 points) Show that the transfer function G(s) from x(t) to θ(t) takes the following form, where $\tau_L = \sqrt{l/g}$

$$G(s) = \frac{\Theta(s)}{X(s)} = \frac{-s^2/g}{(\tau_L s + 1)(\tau_L s - 1)}$$

   c. (10 points) What are the poles and zeros of the system? Is this system stable?

2) Actuator Modelling:
   The system is driving with an electrical motor at voltage input *v*(t), as following:

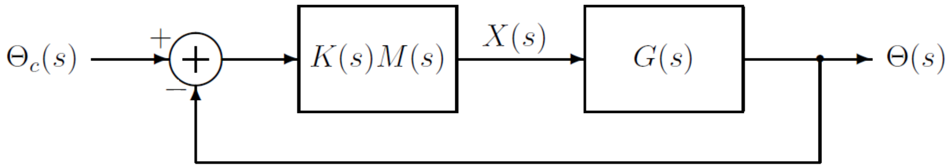


The motor has an internal resistance R_M and a coil inductance L_M. The inductance L_M is typically small and can be ignored. The motor produces a torque proportional to the electrical current *i*: $T = k_t\, i$. The rotation of the motor creates a back electromotive voltage $e = k_e \dot{\varphi}$ in the circuit, where k_t and k_e are torque and voltage constants. The actuation displacement $x = R\varphi$.

(20 points) Please write the electrical circuit equation with v(t) as input, mechanical motion equation with $T = k_t\, i$ as input, and show that the transfer function from *v*(t) to *x*(t) will take the following form:

$$M(s) = \frac{X(s)}{V(s)} = \frac{k_M}{s(\tau_M s + 1)}$$

3) (You may still continue working on Problem 3 even if you didn't figure out the Problem 2). Assume we measure the angle θ(t) of inverted pendulum, and use the motor to control pendulum to follow the command $\theta_c(t)$. With the controller K(s), the block diagram of the compensated system will be in the following.



a. (5 points) What the open loop function L(s)?
b. (20 points) What are the poles and zeros of the loop function for a Proportional Controller $K(s)=K_P$? Will the Proportional Controller $K(s)=K_P$ be able to stabilize the system? Please draw the root locus and justify it. Assume $\tau_L = 1$ second and $\tau_M = 0.5$ seconds.
c. (20 points) Some engineer proposes to use a proportional-integral (PI) controller to stabilize the system,

$$K(s) = K_P + K_I \frac{1}{s} = K_P \frac{\tau_K s + 1}{\tau_K s}$$

Please draw the root locus and justify the feasibility by assuming $\tau_K = \frac{2}{3}$, $\tau_L = 1$ and $\tau_M = 0.5$.
d. (5 points) In the closed-loop system with the above PI control, how many pole and zero cancellations you observe? What might be the potential risk of pole and zero cancellation?